Mathematics Education Research Journal 2010, Vol. 22, No. 1, 91-118

Expanding Participation in Problem Solving in a Diverse Middle School Mathematics Classroom

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In this paper, we discuss our experiences with an after-school program in which we engaged middle-school students with low socioeconomic status from an urban community in mathematical problem solving. We document that these students participated in many aspects of problem solving, including the posing of problems, constructing justifications, developing and implementing problem-solving heuristics and strategies, and understanding and evaluating the solutions of others. We then delineate what aspects of our environment encouraged the students to take part in these activities, particularly emphasising the proactive role of the teacher, the tasks the students completed, and the social norms of our after-school sessions. Finally, we discuss the relationship between our study and the literature on equity research in mathematics education.

The United States, like nearly all developed countries in the world, struggles with the issue of how to achieve equity in the mathematics classroom. Numerous studies indicate that students with low socioeconomic status (from hereon, low-SES) perform below the national average on standardised mathematics assessments and are more likely to drop out of school early (e.g., Reyes & Stanic, 1988; Madison & Hart, 1990; Miller, 1995; National Commission on Mathematics and Science Teaching for the 21st Century, 2000; National Science Foundation, 2000). As Moses (2001) notes, coursework in mathematics has traditionally been a gateway to technological literacy and access to higher education. Students who fail to master mathematics courses such as algebra are denied access to college and, sometimes, even a high school diploma. The struggles of low SES students in their mathematics courses represent both a social and pedagogical problem for all nations, including Australia.

New Paradigms for Equity Research

In recent years, there has been a shift in the nature of equity research in mathematics education. Traditionally, this research had been strongly influenced by paradigms from cognitive psychology and had taken a deficit approach to understanding the poor mathematical achievement of socially and economically disadvantaged students (Gutierrez, 2002, 2008; Nasir & Cobb, 2002). In other words, researchers have sought to explain mathematical achievement gaps between different groups of students by delineating what knowledge, understanding, and cognitive skills students from various economic, racial, and gender groups tended to lack. Although research using a deficit paradigm has yielded useful findings (e.g., Griffin, Case, & Siegler, 1994), several researchers argue that it also has inherent limitations (e.g., Gutierrez, 2002, 2008; Nasir & Cobb; Schoenfeld). For instance, if one believes that a specific group of students fails to learn mathematics because they lack certain cognitive or linguistic skills, this may abdicate the teacher of some responsibility for these students' learning. That is, some may feel that if a particular group of students do not possess the knowledge to participate in an in-class activity, the teacher cannot be expected to teach these students and it is the responsibility of the school to provide the students in question with instruction outside of the mathematics class to rectify these deficiencies (Schoenfeld).

Based on the pioneering work of Delpit (1988) and Secada (1989), equity research has moved away from the use of deficit models and instead has focused on increasing the opportunities for different groups of students to participate in classroom mathematical activities (e.g., Boaler, 2002; Cobb & Hodge, 2002; Lubienski, 2002; Nasir & Cobb, 2002; Powell, 1986; Schoenfeld, 2002;Wang & Goldschmidt, 1989). Within this paradigm, students with low SES are not seen as failing to learn mathematics because they lack the cognitive capacity or mathematical background to do so. Instead, students from traditionally marginalised groups are not provided with the opportunity to participate in the same types of mathematical activities of other students. Consequently, improving the mathematics education of these students does not consist of compensating for the perceived shortcomings of these students, but rather involves increasing their opportunities to learn mathematics by expanding their participation in mathematical activity (Diversity in Mathematics Education Center for Learning and Teaching (DiME), 2007).

Students' Mathematical Autonomy

In this paper, we discuss our experiences working with middle-school students from a poor community in an innovative after-school mathematics program. One significant goal of this program was to provide students with opportunities to exercise their *mathematical autonomy*, where a *mathematically autonomous* student is one who is aware of and relies upon his or her own mathematical resources when making mathematical decisions or judgments (Kamii, 1985).[6] The main purpose of our first six meetings with the students was to negotiate social norms that would be conducive toward achieving these goals (cf., Yackel & Cobb, 1996; Yackel & Rasmussen, 2002). In particular, we invited students to take responsibility for aspects of mathematical problem solving that were typically reserved for the teacher, including deciding if a solution to a problem was correct, judging whether an explanation made sense, and posing directions for future investigation.

Theoretical Perspective and Research Questions

We believe that students' participation in mathematical activities is dependent upon three interrelated factors: (a) the nature of the tasks that students are asked to complete, (b) students' expectations and beliefs about their roles as mathematical learners, and (c) the social norms of the students' environment. We elaborate on each of these factors below.

Before doing so, we note that another overarching factor influencing students' participation is the societal forces that help to shape students' views of themselves as mathematics learners (DiME, 2007). Societal and cultural issues, including similarities and differences in the practices and language used in students' mathematics classrooms and out-of-school activity (Cobb & Hodge, 2002; Hand, 2003), the perceived relevance of mathematics to students' lives (Nasir, 2002), and the labels and discourses used to describe the engagement with mathematics for students of colour (Martin, 2003), can impact students' mathematical identity and the extent to which they will capitalise on the opportunities to participate in mathematical activity (DiME, 2007). These larger societal and cultural issues did not play a central role in our classroom design or data analysis, but we will return to these issues in the discussion section when we discuss the implications of our work with equity research.

Mathematical Tasks

The way a mathematical task is structured may invite students to participate in some aspects of problem solving while preventing them from

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[6] We concur with Yackel and Cobb (1996) that mathematical autonomy is *not* a context-free characteristic of an individual student. Whether a student is mathematically autonomous or not depends upon the way that student interacts with his or her classroom environment, where the established norms of the environment are particularly important.

participating in other aspects. In traditional classrooms, students are usually given exercises that can be completed by applying a procedure that they have just been shown how to use (e.g., Schoenfeld, 1985). For these tasks, students are participating only in a limited aspect of problem solving – implementing an algorithm – and do not have the opportunity to engage in other aspects of problem solving, such as representing the problem, creating and applying heuristics, and determining whether an answer is correct. On the other hand, thoughtful tasks can elicit desirable forms of student engagement. Complex, open-ended problems can encourage students to create novel representations and develop and implement powerful problem-solving strategies[7] (Maher, 2002; Francisco & Maher, 2005). Mathematical tasks that seem to have two plausible but conflicting solutions or tasks that encourage generalisation naturally promote justification and sense-making (e.g. Ellis, 2007; Rasmussen & Marongelle, 2006; Zaslavsky, 2005). Finally, it is important to note that the opportunities that students have to engage with a particular task depend upon the conditions of the classroom, potentially going beyond how the task is written. When certain desirable classroom conditions are in place, such as allowing students sufficient time to explore a task and giving students opportunities to revisit tasks and reflect on their prior work, students are more likely to create novel representations and construct justifications (Maher, 2002; Maher & Martino, 1996). It is important to note that low-SES classrooms tend to be highly structured and thus may deny students the opportunity to participate in the aspects of problem solving described above (cf., Powell, 2004).

Student Expectations and Beliefs about their Roles as Mathematical Learners

Students and teachers enter mathematical courses with implicit understandings of what their responsibilities are and how they are expected to behave; these implicit understandings are sometimes said to be part of a didactical contract (cf. Brousseau, 1997). If students are given a task that violates that contract, they may not engage with the task in the way that the teacher intends or they may refuse to participate in the task altogether (e.g., Herbst, 2003; Herbst & Brach, 2006). To illustrate, Schoenfeld (1985) observes that many students believe that they are incapable of generating new mathematics and should not be expected to do so. As a result, they will not

[7] Of course, what constitutes a complex problem that encourages the creation of novel representations is dependent upon students' mathematical knowledge, illustrating that the nature of students' mathematical knowledge can also influence their participation.

create new representations or heuristics in their problem solving, even when the problem they are solving invites them to do so. Citing her own classroom experiences, Lubienski (2000, 2002) warns that students with different SES may enter mathematics classrooms with different expectations, implying that the same task might evoke different patterns of participation in different groups of students.

Classroom norms. Research suggests that there is a strong relationship between students' mathematical beliefs and the social aspects of their mathematical environment (Cobb & Yackel, 1996; Yackel & Cobb, 1996; Yackel & Rasmussen, 2002). To characterise this relationship, Yackel and her colleagues refer to social norms as regularities in the social interaction patterns of a classroom. Cobb and Yackel (1996) argue that the norms of a classroom environment and individual students' mathematical beliefs are deeply intertwined. In particular, individuals' beliefs can be thought of as their understanding of the normative expectancies of their mathematical environment. Likewise, social norms can be thought of as taken-as-shared beliefs that make possible smooth communication in classroom interactions (Cobb, Yackel, & Wood, 1993; Yackel & Rasmussen, 2002).

Following Yackel and Cobb (1996), we argue that the norms of a mathematics classroom may either invite or inhibit students' expression of their mathematical autonomy. Students are more likely to exercise their mathematical autonomy in situations where they perceive it appropriate to make contributions using their own mathematical resources. Mathematics classrooms with productive norms and in which students regularly take responsibility for deciding what mathematics should be investigated and whether a mathematical idea or argument makes sense will lead to more opportunities for students to express their mathematical autonomy.

Research Questions

This paper has two goals. Our first goal is to document the success of our after-school program at creating mathematically autonomous students who expanded the ways in which they participated in mathematical problem solving. We will illustrate how the students regularly posed challenge problems for each other to solve, justified their solutions to problems, attended carefully to each other's justifications, and challenged and amended these justifications. Our second goal is to examine what aspects of our after-school environment made this increased participation possible. In particular, we will address the following three related questions:

- How did the tasks chosen by the researchers provide students with the opportunity to expand their participation in mathematical problem solving?

- How did the social norms of the after-school environment encourage students to take more responsibility in their problem-solving activity?
- What were the researchers' roles in negotiating these productive norms?

Research Context and Methods

Research Setting

The research reported in this paper occurred in the context of the "Informal Mathematical Learning" research project.[8] In this project, an innovative after-school program was implemented at Hubbard Middle School in Plainfield, New Jersey. Plainfield is an economically depressed area; 74% of the students at Hubbard Middle School participate in a free or reduced-price lunch program. The students at Hubbard Middle School are predominantly students of colour; 98% of the students are African American or Latino. At the beginning of the 2003-2004 academic year, our research team made a brief presentation at a parent-teacher conference in which we invited 6th-grade students to participate in the Informal Mathematical Learning program. Twenty-four 6th-grade students, all African American or Latino, volunteered to participate. These students were representative of the student population at Hubbard in terms of grades, scores on standardised tests, and SES. In fact, the participants in this study had slightly lower grades and standardised test scores than their classmates who did not participate.

In the after-school sessions, students were asked to work on open-ended, well-defined mathematical problems. In general, these problems were of a type that the students had not seen before. The complete list of problems in which the students engaged are presented in the Appendix. The researchers encouraged collaboration among students, frequently asking them to work together on tasks. Justification was also encouraged and students were frequently asked to explain their solutions to their peers. At the same time, the students were never told whether their reasoning or solutions to problems were correct. The students themselves were expected to be the arbiters of which explanations made sense and were acceptable. All sessions were videotaped. The goals of this research study were to understand how students' mathematical reasoning developed in this

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[8] The "Informal Mathematical Learning" Project, led by Carolyn Maher, Arthur Powell, and Keith Weber, is supported by the National Science Foundation ROLE Grant REC0309062.

problem-solving environment over time and to investigate the relationship between mathematical autonomy and mathematical reasoning.

When organising these sessions, we did not directly consider the SES of the participating students. In fact, much of our environment was consistent with how we organised informal learning environments with other groups of students (cf., Maher, 2002). We did not want to begin our engagement with students with preconceived notions about their abilities or dispositions. To avoid such stereotypes, we treated students as individuals with unique life histories and experiences (cf., Gutierrez, 2002). However, we were aware that students entered our program with life histories that impinged (positively and negatively) on how they interacted with us, with each other, and with the mathematical tasks that we presented to them and they presented to each other. Like classroom teachers, we did not have access to these life histories. As such, we carefully attended to the meaning that students attached to their words and actions and the way students seemed to interpret our words and actions. We used insights gained from this attention in formulating tasks for students to work on and negotiating what constitutes acceptable researcher-student and student-student interaction patterns. Later in the paper, we will discuss how our study relates to the mathematics education research literature on equity.

Data Collection

The study was a longitudinal one, spanning 3 years. This paper will present the initial stages of analysis, focusing on the first 3 weeks of the study. The decision to focus on these beginning sessions was based on the fact that we wanted to see how productive classroom norms were established and investigate the extent to which students participated in an increasingly wide range of problem-solving activity. Analysing later sessions, in which some classroom norms were already established and student expectations were already in place, would not be appropriate for achieving these goals. The participants met with the researchers twice a week, with each meeting lasting approximately 75 minutes. There were six meetings during the 3-week period, totalling approximately 8 hours. During the sessions, three cameras videotaped these classes. One camera focused on the teacher-researcher while the students were completing their investigations and the students at the overhead projector when the students were sharing their work with the class. Each of the other two cameras focused on an individual table of three or four students. These students served as focus groups for our analysis. During these meetings, students were primarily engaged in problems about fractions using Cuisenaire rods. A question representative of those that students were asked to solve is "If I gave the light green rod the number name one, what number name would I give to the yellow rod?" Each of these lessons was videotaped, yielding a total of approximately 24 hours of video to be analysed.

Data Analysis

Initially, three authors of this paper independently viewed the videotapes and documented every episode in which we felt a student was behaving in a mathematically autonomous manner (i.e., relying on his or her own mathematical resources, rather than cues from a classmate or authority figure, in making a mathematical judgment). There was a high level of agreement on the identified instances, and any disagreement was discussed and used for further refinement of the authors' understanding of the notion of mathematical autonomy. For each episode, the authors then identified the aspects of the environment, with special attention to the task the students were completing and the norms of the environment, that encouraged students to exercise their mathematical autonomy. Finally, when we observed productive social norms that invited student participation, we focused our attention on episodes in which these norms were negotiated between the students and the researcher.

Results—Selected Episodes

In this section, we present a qualitative account of how productive norms were established in our classroom environment, how these norms invited students to participate in many aspects of problem solving, and how this participation enabled students to reason in sophisticated ways. In the next section, we will present quantitative evidence that the episodes that we are presenting here are indicative of what transpired in the sessions that we analysed.

Episode 1

The first episode occurred early in the first class meeting. The 24 students in the class were partitioned into seven tables; each table contained a group of three or four students. There was a bag of Cuisenaire rods placed in the middle of each table. Students were asked to examine the Cuisenaire rods and make observations of them. When a student made an observation, he or she was asked to come up to the overheard and share it with the class. The following excerpt occurred in this context:

Researcher 1: Shanae, do you have a comment?

Shanae: Two yellows make a whole [by "a whole", Shanae is referring to an orange rod, the longest of all the Cuisenaire rods].

R1: Did everyone hear what Shane said? Can you come up here and show us what you just told us?

R2: Not everyone heard you Shanae. Come up to the overhead and show us, we have more rods here.
R1: Yeah, you need to speak louder.
Shanae: Two yellows make one whole.
R2: Do you want to come up and show us? There's room over there to show us.
[On the overhead, Shanae lines up two yellows rods and shows that these have the same length as one orange rod]
R1: Isn't that interesting? Did you see what Shanae just did? Okay, I want you all to think about what Shanae just did, and she said that two yellows make one whole. I have a question […] I am going to ask it to Shanae, but I am going to ask it to all of you. If I gave the yellow rod the number name five, what number name would I give to the orange rod, raise your hand if you think you know. If you know discuss it at your table and see if you agree.
Dante: I think it would be ten because…
Chanel: Half of the orange is yellow.
R1: Okay, discuss it at your table, if everyone at the table agrees, raise your hand. If you haven't discussed it at your table, I really want you to discuss it at your table.

This excerpt is important since the episode began to establish regularities in the classroom that would be prevalent throughout the analysed episodes. At the most basic level, when students had a finding, they were asked to share their finding with the class and explain or justify their finding, often while using the overhead projector to display their work. The teachers also expected the students to carefully attend to one another's work. There were more sophisticated interaction patterns being introduced here. First, Shanae's observations served as the basis for the next task posed by the researcher. This was the first instance of a student's contribution steering the subsequent investigations of the class, a theme that would become increasingly common as the sessions progressed. Second, after the first researcher posed the question, one student correctly claimed that the answer was 10 with a second student offering a justification for this solution. However, the researcher did not acknowledge these students' comments; instead she insisted that the students first discuss the work at the table and only raise their hand when a consensus at the table was reached. We will illustrate in later excerpts how, when this interaction pattern became normative, students tried to understand one another's explanation, took an interest in their classmates' reasoning, and shared responsibility for deciding which solutions made sense.

Episode 2

Like the previous episode, this episode also takes place in the first session and is presented here since it illustrates the development of important norms that were present throughout the class sessions.

R1: I'm going to call the orange rod one, I want to know what number name we would give to the yellow rod, discuss it at your tables.

Chris: [at his table] 0.5. 0.5 times two is one, right?

Jeffrey: [to Chris] You can't really do a division with one divided by two so you do .5 times two.

R1: Raise your hand if you're ready to discuss it […] I'm asking you what the number name is for yellow and I'd like to hear from Chris.

Chris: Um, the number 0.5.

R1: You're going to call it 0.5, how many people said 0.5? Only a couple of people. Why Chris?

Chris: Because if you add 0.5, since it's a decimal, if you add five plus five regular it would equal ten but it's a decimal so if you add it with a decimal, it would equal to a whole number which is one.

R1: So if you add 0.5 and 0.5 you get one. How many people think that's reasonable what Chris just said, did anyone think something else? … I want to hear from Malika, okay Malika, I want a number for yellow and all I know is the orange rod is one. What are you telling me that yellow has what number name?

Malika: One.

R1: One? Malika says the number name one, what do you all think about that? […] Do you think that's okay? Dante agrees. Okay now, why? Why does it work? How can you show it has the number name one? Um, what do you think, Chris?

Chris: You can't do it because if you put one you're not really dealing with multiplication so if you do one and one it will equal two, but the orange rod is only one.

R1: So what you're saying is, what I'm hearing you say, that if the length of the yellow rod is one and the length of the other yellow rod is one, that the length of the yellow rod along with the yellow rod which is the orange rod would have to be two, but Malika thinks they both can have the number name one. Kendra?

Kendra: I think it might be a half because a half and a half is one whole […]

R1: So some of you I think here were thinking that you're multiplying and some of you are thinking that you're adding… What makes sense here? Should we allow multiplying and adding? What are we paying attention to? What do you think?

This conversation continued with another student suggesting the answer might be 1.5. No consensus among the students was reached before a new question was posed to the students. There are several characteristics about this excerpt that became regularities in subsequent classroom interactions. First, at no point did the researcher tell a student if he or she was correct or incorrect. Rather, she tried to deflect responsibility in establishing the reasonableness of an answer or explanation back to the students by continually asking students what they thought and whether or not an explanation made sense. Second, the researcher tried to elicit as many different answers as she could from students. As Rasmussen and Marongelle (2006) argue, eliciting several alternative solutions naturally helps students feel the need to justify, so as to decide which solution is correct. Finally, this sequence ended without a consensus being reached, a sharp violation of traditional classroom discourse where a student offers a solution and the teacher evaluates it (Mehan, 1979). It is worth noting that several classroom regularities are already beginning to form. Chris and Jeffrey are observed discussing their solution to the problem with each other before offering it to the class, while Chris and Kendra both accompany their solutions with a justification.

Episode 3

In the following episode, students were grappling with the question: "If the blue rod had the number name one, what do you call the white rod?" Chanel was able to deduce the answer was one ninth, since nine white rods were as long as one blue rod, but Dante, her tablemate, was unable to arrive at an answer. The exchange below illustrates Chanel helping Dante:

Chanel: [addressing Dante at their table]If we take this one [pointing to the blue rod], that's a whole…and you take one of these [taking a white rod and placing it along the blue one], it's 1.9 [NOTE: *Chanel and Dante use 1.9, 1.3, and 1.5 to refer to one ninth, one third, and one fifth, respectively*] …so if you take some more of these, that's 1.9+1.9+1.9+…+1.9 [9 times]. Take the white ones away…[places three light green rods along the blue]. Now, what is this called now? [referring to light green]

Dante: 1.3

Chanel: Why?

Dante: It's 1.3 because 3 light green rods make up the blue … [*happily*] I'm a genius!

We argue that the norms established in the first two episodes are influencing Chanel's behaviour. At this point, it has been established that solutions should make sense to students and that students are encouraged to share their solutions with the class only after everybody at their table agrees with the solution. A psychological correlate of these norms is that students are now responsible for the mathematical understanding of their tablemates. In the excerpt above, Chanel accepts this responsibility. She first offers an explanation for why the white rod should have the number name one ninth.

(Chanel and Dante mistakenly say 1.9, 1.3, and 1.5 instead of one ninth, one third, and one fifth, respectively, but their reasoning is otherwise correct[9]). However, she is not satisfied that Dante heard her explanation. She attempted to see if he understood the explanation by seeing if he could apply similar reasoning to another example and then provide his own justification for why his solution was correct. The cognitive benefits of Chanel's tutoring are easy to observe: After Chanel's intervention, Dante appears to understand a mode of reasoning that he had not previously demonstrated and Dante appears happy with his newfound understanding.

Episode 4

The following excerpt occurs while students are discussing their solutions to the problem, "If the blue rod has the number name 1, what number name should we give to the light green rod?" There was a debate as to whether the light green rod should be assigned the number name one third or 0.3. (Note that three light green rods have the same length as one blue rod). During this debate, Dante asks what number name the white rod would have if the light green rod had the number name 0.3:

Dante: Since 3 white cubes go into a light green rod, what are we gonna call …if we call the light green 0.3, what are we going to call the white rod?

R1: Let's follow this here, what Dante says. […] He said if 3 white rods are the same length as light green…Do you all agree with that?

Class: Yes.

R1: So, the question is what are we going to call the white rod. […] I want you all at your tables to figure that out.

We feel this episode is significant because Dante is pushing the boundaries of what responsibilities are usually assigned to students in mathematics classrooms. In many classrooms, the teacher's goal is to ascertain what the students know. He or she therefore has the responsibility of posing questions to students and evaluating their answers. The students have the responsibility of presenting their mathematical knowledge for evaluation. In such environments, students are not expected to pose the questions and, indeed, may refuse to participate in activities that require them to do so (e.g., Herbst & Brach, 2006). In the presented episode, the researchers received Dante's contribution positively. They first called

[9] We do not know why Chanel and Dante used these idiosyncratic representations for fractions and acknowledge this could significantly hinder their understanding of decimals in the future. The researcher later attempted to resolve the issue by asking students to discuss what the white rod should be called.

students' attention to what Dante was saying and then asked the students to solve Dante's task. The researchers implicitly endorse Dante's behavior by calling students' attention to it. In short, this episode illustrates Dante and the researchers negotiating a new pattern of interaction – the students themselves may take responsibility for posing problems to their classmates. Dante's previous experiences with Chanel (see episode 3) and the norm that students' participation would influence subsequent investigations may have encouraged Dante to pose this problem to his classmates. In episode 6, we will illustrate how one problem posed by a student expanded the ways his classmates thought about fractions.

Episode 5

In this episode, a group of students were working on the problem, "If five rorange trains had the number name one, what would the number name for red be?" (A "rorange train" was the name assigned by the students for an orange rod and red rod laid end-to-end.) The students initially struggled with this question, with different students conjecturing that the number name for red was 1/2, 1/6, 1/15, and 1/30 (the correct answer). During this debate, the following discussion took place:

Jeffrey: Because we call this whole line one [pointing to the five orange and five red rods lined up side by side] and it takes 30 red rods to cover the whole thing so it has to be one thirtieth. [Malika, working independently, can be heard in the background saying, "It is one sixth"]

R3: So you're saying that because… say that again, I'm sorry.

Jeffrey: Because we're calling this whole line one whole and it takes 30 red rods to complete it, we should call it one thirtieth.

R3: So Jeffery's saying one thirtieth and Malika is saying one sixth…

Malika: One-sixth.

R3: So did you hear Jeffrey's explanation? So he said it's because… [Jeffrey is grinning widely as he taps the researcher on the arm and points to a sign on the wall that says "Prove it"]

R3: [to Jeffrey] What are you looking at there? [Jeffrey continues to grin as he points to the sign]

R3: Oh, yeah. Prove it! That's a good question. Malika, show to me that it's one sixth.

Methodologically, an important way to determine if norms are present in a classroom environment is to observe students' reactions when they are transgressed. When Jeffrey presents his answer of 1/30, he presents a justification for why his answer is correct. It is notable that when Malika arrives at a different answer, Jeffrey hints to the researcher that Malika should be asked to justify her solution. We would argue that the norm that solutions should be accompanied by justifications (especially when students arrive at different solutions) had convinced Jeffrey that it was his and other students' responsibility to construct arguments to support their assertions (cf., Cobb & Yackel, 1996). This excerpt also illustrates Jeffrey negotiating a more active role in this classroom environment. Prior to this point, the request for justifications usually came from the researchers. One interpretation of Jeffrey's behavior was that he did not feel it was his place to ask for a justification but it was the researcher's responsibility to do so.

Episode 6

In the final excerpt, students were told that the combined length of an orange and red Cuisenaire rods, placed end-to-end, had the number name one. Students were then asked to determine what number name the white, red, light green, and dark green rods should have. The students chose to call the orange and red Cuisenaire rods a "rorange train". Four students, David, Nia, Jelani, and Jerel, came to the overhead projector to present their solution to this problem. After presenting their correct solution and explaining how their solution was constructed, Jerel posed a challenge problem to the class:

Jerel: Alright. If rorange is one, what is the yellow rod? [There is a general din of the students mumbling]
R1: Say it one more time, nice and loud.
Jerel: If rorange is one, what is the yellow rod?
Student: [from the back] One half.
R2: Did everyone hear this? If you heard it, talk with your group. If not, the question is… [Many students are heard discussing the problem]
R1: OK. Thank you very much. Very nice challenge!

Prior to this excerpt, it had already been established that students were welcomed to contribute by proposing challenges. In this excerpt, the researchers once again support Jerel's challenge; first by calling everyone's attention to Jerel, second by making sure everyone heard what was being asked, and third by calling Jerel's challenge "very nice." The seemingly taken-as-shared expectation that students were permitted to pose challenge problems to other students probably invited Jerel to present his problem. However, the initial task itself may have also contributed to Jerel's problem posing. Jerel and his classmates were asked to find the lengths of five different coloured rods. We have found that such questions invite generalisations, both in terms of finding a general method to solve all cases and in considering how one would solve cases that were not posed in the original question.

This episode provides an illustration of how having students participate in the activity of posing challenge problems provides them with an opportunity to improve their mathematical understanding. By the point in which the above excerpt occurred, most students had developed what Steffe (2003) calls a partitive fractional scheme. That is, they could recognise, for instance, that the red rod was one fifth of the orange rod because the length of five red rods was as long as the orange rod. Jerel's challenge problem was particularly interesting because it could not be solved by a partitive fractional scheme, a scheme that most students had mastered by this point. The yellow rod did not divide the "rorange train" evenly (the yellow rod had the length of five white rods while the rorange train had the length of twelve). To address Jerel's challenge, most students developed and used what Steffe (2003) refers to as a unit fractional composition scheme. They reasoned that since the white rod was 1/12 and the yellow rod contained five white rods, the yellow rod should have length 5/12. Most students were then able to use this scheme to solve other similar problems. Hence, Jerel's challenge problem provided students with the opportunity to reason about fractions in a more sophisticated way.

Patterns of Student Participation

In this section, we focus on three aspects of problem solving in which students do not traditionally participate when placed in traditional environments: (a) trying to understand the arguments of others, (b) correcting the arguments of others, and (c) posing problems for themselves and others to investigate. In analysing our data, we noticed that in the first few sessions, students rarely participated in these activities. However, students' participation increased dramatically in the subsequent sessions. To document these trends, we analysed the behaviour of two focus groups of students – one focus group was a table with four students and the other group was a table with three students. We chose to study these students because there were cameras documenting these students' behaviours most of the time. For each of the first six class sessions, we counted the number of times that each of the following occurred:

(a) Students questioned the arguments of others. Such questions occurred in situations in which students inquired to seek verification that they understood the argument of their classmate or situations in which the students asked questions to seek information about aspects of others' arguments that were unclear to them. We took this type of questioning as evidence that students were attending to and trying to understand the arguments of their classmates.

(b) Students corrected the arguments of others. When one student presented an argument and an aspect of that argument was faulty, that student's tablemates would sometimes point out and correct that aspect of the argument. We took this as evidence that students were evaluating the validity of and building upon the arguments of their classmates.

(c) Students posed challenge problems for their classmates to solve. We took these instances as evidence that students were actively shaping the direction of the mathematical investigations.

The results of this analysis are presented in Table 1. As the table illustrates, students' participation in each of these activities rapidly increased. For instance, students did not pose a challenge problem in the first two sessions, but posed nine challenge problems over the next four sessions. There did seem to be a slight decrease in participation in the sixth session. This is likely because the questions the students were addressing were somewhat more difficult and they were working collaboratively on a larger task that consumed the majority of the session. As a result, students spent more time generating solutions to problems and less time presenting their solutions to others or posing new problems to be solved.

These findings weaken the plausibility of the claim that the students in our environment participated in the way that they did out of habit or convention. If such participation patterns were common in the students' other mathematical environments, we would have expected more frequent participation in the initial sessions. A plausible alternative hypothesis to our data is that students' participation increased because they become more comfortable and accustomed within our environment. However, teachers at Hubbard Middle School who assisted us with this project emphasised to us that the interaction patterns were not typical of what transpired in the mathematics classes at their schools. The mathematics classes at their schools were highly structured and presented little opportunity for students to solve challenging problems, present justifications, question one another, or pose challenge problems.

Table 1.
Number of Instances that Students Participated in Problem-Solving Activities

Session	Student inquiring about the arguments of a classmate	Student correcting the argument of a classmate	Student posing a challenge problem for a classmate
1	1	0	0
2	8	2	0
3	17	5	2
4	19	14	5
5	27	19	1
6	13	10	1

Discussion

Summary of Results

Increased participation and changes in students' expectations. We documented that students participated in many aspects of problem solving, including posing problems to be solved, generating heuristics and notation to solve the problems, constructing justifications to their solutions, helping other students understand these justifications, and attending to, questioning, and evaluating the justifications of others. As section 5 illustrates, much of this participation was largely absent in the first two sessions of our study, but became increasingly prevalent as our study progressed.

In several of the episodes, we found evidence that students' expectations of their roles as mathematical learners had shifted as our study progressed. For instance, in episode 3, Chanel acted as if it was her responsibility to make sure that Dante understood why the white rod should have the number name one ninth if the blue rod had the number name one. In episode 5, Jeffrey's actions imply he has the expectation that students' answers should be accompanied by justifications and, further, when they are not, the researcher should request a justification. Prior to Dante's challenge in episode 4, students had not posed challenge problems. However, as section 5 illustrates, many students felt comfortable doing this in subsequent sessions.

The nature of the tasks. There were several aspects of the tasks that we provided that may have encouraged students to take a more active role in the problem-solving process. First, the tasks that students were given were not similar to problems that they had seen before and students generally were not aware of a procedure that they could use to solve the problems. In these cases, in order to make progress on the assigned problems, they had no choice but to rely on their own mathematical resources. Second, the tasks were open-ended, in the sense that they either allowed multiple solutions, or multiple ways to arrive at a solution, or both. Such tasks encouraged student creativity. Further, when students presented multiple solutions to a problem, this created an intellectual need for students to justify their solutions in order to determine which solution was correct (e.g., Rasmussen & Marongelle, 2006; Zaslavsky, 2005). For instance, in episode 5, we see how conflicting answers prompted Jeffrey to desire a justification from Malika.

Third, many of the tasks invited students to generalise. For instance, in the first session, students were asked, "If the white rod has the number name 2, what number name would I give to the other nine rods?" Such questions encourage students to come up with a general method for determining the length of a given rod rather than solving nine separate problems. Generalisations can encourage students to provide justifications to show that their solutions are correct (e.g., Ellis, 2007) and to pose extension problems, as illustrated in episode 6.

Classroom norms. A number of classroom norms were established in our initial sessions that may have been useful in encouraging students' participation. First, the researchers received all participation positively but never told students whether they were correct. This may have reduced students' fears about having their mathematics judged by authority figures – a concern that Lubienski (2000) raised based on her own classroom-based research – and allowed them to participate more freely. This also may have encouraged students to take responsibility for determining whether or not a solution was correct or a justification made sense. Second, students were always asked to justify their answers and the acceptability of their justifications was used to decide which answer was correct among multiple alternatives. Third, students frequently presented their work on the overhead projector. When a student was presenting, the researchers would make sure the rest of the class was quiet and attending to that student's arguments. This may have encouraged students to attend to the arguments that others were presenting. Fourth, students were required to convince their tablemates of the soundness of their solution before presenting their work to the entire class. This likely promoted students to take responsibility for the understanding of the other students at their table, as was observed in episode 3. Finally, students' contributions frequently set the stage for subsequent investigations. This may have led students to take a more proactive role in determining what mathematics was investigated, including posing challenge problems for other students to solve.

Learning opportunities created by increased participation. We believe that in some cases, students' increased participation in problem solving provided them with opportunities to advance their mathematical understanding. In episode 3, Dante initially appeared unable to understand why the white rod should have the number name one ninth when the blue rod had the number name 1. After Chanel acted as a tutor, Dante understood Chanel's solutions and could use similar reasoning to answer other questions. In episode 8, Jerel's challenge problem provided students with the opportunity to develop a more sophisticated scheme for reasoning about fractions. We believe these episodes each constitute specific instances where increased participation had specific cognitive benefits for individual children. In Weber, Maher, Powell, and Lee (2008), we discuss the relationship between students' interaction patterns and learning opportunities in more detail.

Caveats and Directions for Future Research

We believe that there were three aspects of our study that limit the extent to which these results could be generalised to actual classrooms. First, the experience of participating in an after-school enrichment program was novel to students. It is likely that they did not have strong expectations of what their experience would be like. This may have facilitated the establishment of productive social norms. These students likely had more defined expectations for how their interactions in their regular mathematics classes should proceed, making the process of negotiating productive norms more difficult for the teacher. Second, unlike the researchers in this study, teachers in most classrooms are accountable for what the students learn. As a result, teachers may find themselves in a difficult position in deciding whether to devolve responsibility of some aspects of mathematical activity to the students or play a more active role in students' investigations. To illustrate one difficulty that a teacher might have, consider a student who poses a challenge question that is not germane to the topic that the teacher intends to teach. Does the teacher spend valuable class time allowing students to consider this question? Or does the teacher not follow up on the student's question, which may discourage students from proposing investigations in the future? Analysing how teachers can implement the ideas discussed under the constraints of public schooling is beyond the scope of this paper, but would be an interesting topic for future research. Finally, the students who participated in this study were volunteers, or more accurately, were volunteered to participate by their parents. Although these students were demographically similar to their classmates who did not participate, it is possible that these students had other characteristics not shared by their classmates, such as a desire to do mathematics. We do not believe that this was the case, but we have no way to test for this hypothesis.

How this Data relates to our Larger Research Project

This data came from the first eight sessions of a 3-year research project. A comprehensive analysis of the research project is beyond the scope of the paper, but some details about how the project proceeded may help the reader interpret our findings. First, the norms of this study were stable and continued throughout our interactions with the students. An illustration of how some of these norms provided the students with opportunities to engage in sophisticated mathematics is discussed in Weber et al. (2008). Second, after observing our lessons, teachers in the school district in which our study was couched were able to successfully create a problem-solving environment that was similar to what was reported here with another group of students. Third, students who went through these environments showed gains on their standardised tests, suggesting that environments of the type that we created may be influential in closing the achievement gap. The latter two findings will be the subject of future reports.

Relevance to Equity Research

Previous research on students of low SES, especially poor students of colour, suggests that, in many cases, their classrooms are highly structured with a strong emphasis on procedures. Students in these environments are asked to observe and then implement procedures; they are given few opportunities to use their own mathematical resources in making decisions. Perhaps consequently, many such students develop highly procedural views of mathematics and engage in mathematical activity only when they perceive it to be necessary (e.g., Martin, 2000; Schoenfeld, 2002; Powell, 2004; DiME, 2007). The results of this study contribute to this research in three ways. First, the population in our study consisted of middle-school students of colour in a poor community. Our study indicates how teachers might successfully engage them in powerful mathematical reasoning, including forming and evaluating justifications, posing interesting challenge problems for investigation, and developing and implementing strategies to solve difficult problems. If these students did not engage in similar activities in their classrooms, it is not because they lacked the cognitive capacity to do so. Second, the students in our study rapidly took responsibility for their own mathematical learning and activities, as well as the learning and activities of their classmates. This reinforces Yackel and Cobb's (1996) argument that the mathematical autonomy of students is not a context-free trait of an individual student, but rather a characteristic of how that student interacts in his or her mathematical environment. Within our environment, the students in our study exercised their mathematical autonomy in a relatively short period of time. Third, social norms and taken-as-shared expectancies were established within the first three meetings of our study. Previous research suggests that productive norms can be established quickly in mathematical environments and that these norms can have a positive effect on students' mathematical activity (Yackel & Rasmussen, 2002). Our study replicates these results in a different environment.

The socioeconomic status of our students did not play a major role in how we created our classroom environment. We preferred to engage students without preconceived notions of what their dispositions were or how they would behave. However, in our analysis, we did find five aspects of our environment that might be particularly beneficial when working with students with low SES. First, the researchers refrained from evaluating the correctness or sophistication of the students' mathematical reasoning. Previous research suggests that students with low SES may be less likely to participate in classroom mathematical activities out of fear of being judged negatively by the teacher or the perception of the teachers' stance toward them (e.g., Delpit, 1988; Frankenstein & Powell, 1989; Lubienski, 2000, 2002). The lack of researcher judgment in our study may have encouraged more participation. Second, the researchers strove to give students meaningful tasks that were not placed in a pseudo-real world context, tasks that other researchers have suggested might cause confusion with this population of students (Lubienski, 2000). Further, the researchers would negotiate the meaning of a task before students began investigating it, something that Boaler (2002) argues can reduce linguistic, class, and ethnic inequalities in inquiry-based instruction. Third, although some participatory norms were strongly encouraged by our research team (e.g., working collaboratively and justifying one's solutions), students had an active role in negotiating many aspects of their participatory roles (for example, see episodes 4 and 5). As Hand (2003) notes, "open" participation practices – that is, allowing students to negotiate their own roles in their mathematical environments – lead students to become more invested in their mathematical activity than if an authority tells them how to behave. Fourth, we permitted and encouraged students to use their own language and modes of reasoning to justify their solutions and discuss their mathematical work. Barriers to student participation in mathematical activity include discontinuities between their language and reasoning in out-of-school activity and their mathematics classrooms (Cobb & Hodge, 2002; Hand, 2003). Allowing students to use their own language and reasoning not only increased their mathematical autonomy, but perhaps also limited such discontinuities.

Our fifth and final aspect is more tentative. In our environment, the researchers displayed a genuine interest in students' mathematical reasoning, often letting students' reasoning set the stage for subsequent investigations. We found that students were initially surprised that authority figures valued their mathematical ideas; it is possible that these students were accustomed to teachers not valuing their mathematical reasoning, but only caring if their answer was correct. In our informal observations, we found that once students realised that we cared deeply about how they thought, they were eager to share their thinking with us.

Acknowledgments

This research was supported by a grant from the National Science Foundation (ROLE Grant REC0309062). The views expressed here are not necessarily those of the National Science Foundation. We would like to thank Chris Rasmussen and Karen Marrongelle for helpful advice in clarifying the theoretical framework used in our study, as well as the anonymous reviewers for helpful comments on earlier drafts of this manuscript.

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Appendix.

Tasks Posed by Researchers

Date	Tasks
11/12/03	1. If I gave the yellow rod the number name five, what number name would I give to the orange rod? 2. Suppose I gave the orange rod the number name four what number name would I give to the yellow rod? 3. If I call the orange rod one, what number name would I give to the yellow rod? 4. If I call the white rod two, what number name would I give to all of the other rods?
11/13/03	1. Suppose I called the dark green one what number name would I give to the light green? 2. Someone told me that the red rod is half as long as the yellow rod, what do you think? 3. If I call the blue rod one, I want each of you to find me a rod that would have a number name one-half.
11/19/03	1. Convince us that there is not a rod that is half the length of the blue rod. 2. Is 0.3 another name for the light green rod? 3. If I call the blue rod one, what number name would I give to the white rod (red rod)?
11/20/03	1. If I call the rod blue rod one, what number name would I give to the red rod (light green rod)? 2. Is three-ninths another name for the light green rod? 3. If I call the blue rod one, what number name would I give to the yellow rod? 4. If I call the blue rod one, what number names would I give to the rest of the rods?
12/3/03	1. If I call the orange rod one, what number name would I give to the white rod (red rod)? 2. Can the red rod be named two-tenths and one-fifth? 3. If I call the orange rod ten (fifty) what number name will I give to the white rod? 4. I want to know which is bigger, one-half or one-third and by how much.
12/4/03	1. Build a train that would be named one if the yellow rod were named one-half. 2. How many distinct trains of two purple and two white rods are there?
12/10/03	1. If we call “rorange” one, what would be call the red rod? 2. If a train of two red rods and two orange rods is called one, what number name would you give to the red rod? 3. If a train of two red rods and two orange rods is called two, what number name would you give the dark green rod? 4. If a train of two red rods and two orange rods is called two, what number name would you give the white rod?

	5. If three "roranges" were named one, what number name would you give the red rod? Four "roranges"? 6. If we call "rorange" one, what would be call the yellow rod? 7. If we call 100 "roranges" one, what number name would you give the red rod? 8. What is the number name for the white rod when ten "roranges" are named one?
12/11/03	1. Suppose I had two "roranges" (a train of red and orange), what number name would I give to the red rod? 2. If ten "roranges" are equal to one, what is the number name for the red rod? 3. If ten "roranges" has the number name one, what would the number name of the white rod be, 120 or 1/120? 4. If six "roranges" had the number name one, what would you call the red rod? 5. Suppose you had a friend and he was given a train with a bunch of red and orange rods. The train was called one, how could he find a name for red? 6. I have a long train of "yoranges" (yellow and orange) called one, what would the yellow rod be called? What if I had five "yoranges"? Six "yoranges"? 7. If a train of twenty orange rods and five red rods is named one, what is the number name of the red rod? 8. Suppose you had a large number of "roranges", how could you name a red rod?

Challenge Tasks Posed by Students

Date	**Challenges**
11/12/03	no student challenges were posed
11/13/03	no student challenges were posed
11/19/03	1. If the light green rod is called 0.3, what is the number name for the white rod? 2. What is the number name for the red rod when the blue rod is called one?
11/20/03	1. If the blue rod is named one, what is the orange rod named? 2. What is the name of the white rod if the black rod is named one? 3. How many rods are equivalent to two green rods? 4. Why are ten white rods equivalent to an orange rod but only nine white rods are equivalent to a blue rod? 5. If the orange rod has the number name one, what is the number name for the yellow rod?
12/3/03	1. If the orange rod is named one-twenty, what is the blue rod called?
12/4/03	1. How many trains can you form that has the same length as the orange rod?
12/10/03	1. If a train of two orange rods and two red rods is named two, what number name would you give to the purple rod?

	2. If a train of two orange rods and two red rods is named two, what number name would you give to the yellow rod? a light green rod? a brown rod? 3. If ten “roranges” are named one, what number name would you give to the red rod and why? 4. If nine orange rods and four red rods are named one, what is the name of a red rod? a black rod? a white rod?
12/11/03	1. If “rorange” (a train of an orange and a red rod) is named one, what is the number name for the yellow rod? 2. If “blorange” (a train of an orange and a blue rod) is named one, what is the number name for the light green rod? 3. If we call ten “roranges” one, what number name would be call the white rod? 4. If ten “yoranges” (a train of an orange rod and a yellow rod) are named one, what will the yellow and light green rods be called? 5. How many white rods are equal to ten “yoranges”? 6. If “rorange” is named one, what is the number name for the yellow rod? 7. If “rorange” is named one, what is the number name for the black rod? 8. How many light green rods are equivalent to ten “yorange” rods? 9. How many white rods are equivalent to ten “yorange” rods? 10. How many yellow rods are equivalent to ten “yorange” rods?